Transient behavior of particle transport in a Brownian motor

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The transient behavior of a Brownian motor is investigated for more detailed particle transport occurring therein. The asymmetric nature of the time-dependent mean particle velocity is examined during the transition between two different levels of thermal noise. The possibility of current inversion is also investigated. It is found that the detailed shape of the asymmetric potential is crucial for such an inversion to occur.

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I. INTRODUCTION

The phenomenon of the Brownian motor, also known as a ratchet, has been investigated intensively in recent years. It is directed particle transport in a system not in thermal equilibrium and with a spatial asymmetry due to an asymmetrical potential, nonequilibrium perturbation, or a special type of coupling. A comprehensive review of the general concept, types of ratchets, fundamental theories, and their applications was given by Reimann [1].

Consider a one-dimensional system governed by the equation

$$\eta \dot{x} = -V'(x) + \xi(t), \tag{1}$$

where $x(t)$ is the location of a particle, η is the friction coefficient, $\xi(t)$ is the thermal noise, $V(x)$ is the potential, and the inertia of the particle is neglected. The thermal noise $\xi(t)$ in Eq. (1) is modeled as a Gaussian white noise with the following correlation function [1,2]:

$$E[\xi(t)\xi(t+\tau)] = 2\eta k_B T(t)\delta(\tau), \tag{2}$$

where $E[\ ]$ denotes an ensemble average, k_B is a constant, and $T(t)$ is the environmental temperature assumed to be periodically varying in time. The potential $V(x)$ is spatially periodic with a period L and is asymmetric within each period L. The functional form of $V(x)$ depends on the physical problem involved. The system governed by Eqs. (1) and (2) is a typical temperature ratchet.

It is known that particle transport in a temperature ratchet is caused by two factors: the asymmetric potential and a time-dependent temperature field. In earlier investigations [1], interest has been focused on the long-term average of the particle current, defined as

$$\langle \dot{x} \rangle = E\left[\lim_{t\to\infty} \frac{x(t)-x(0)}{t}\right] = E\left[\lim_{t\to\infty} \frac{x(t)}{t}\right] \tag{3}$$

where $x(t)$ is the particle location in an unbounded one-dimensional space. Note that both time average and ensemble average are applied in Eq. (3); thus, the initial state has no effect on the result. Equation (3) can also be expressed as

$$\langle \dot{x} \rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t E[\dot{x}(t)]dt. \tag{4}$$

The two averaging procedures, i.e., the ensemble averaging and the time averaging, suppress many details of the particle movement. To better understand the Brownian motor phenomenon, as well as identify and construct different types of ratchets, more detailed knowledge of the particle movement may be required. The present paper is aimed at obtaining a greater insight into the particle motion by carrying out only the ensemble averaging $E[\dot{x}(t)]$, without additional time averaging. In what follows, $E[\dot{x}(t)]$ will be referred to as the transient mean velocity, which is generally time dependent and is a constant only in an invariant temperature field.

For the present temperature ratchet model, the transient mean velocity can be obtained from the governing equation (1) as follows:

$$E[\dot{x}(t)] = -\frac{1}{\eta}E[V'(x)] = -\frac{1}{\eta}\int_{-\infty}^{\infty} V'(x)p(x,t)dx, \tag{5}$$

where $p(x,t)$ is the probability density of the particle displacement $x(t)$. Since the input $\xi(t)$ in Eq. (1) is a nonstationary stochastic process, analytical solutions are obtainable only for their asymptotic forms. In general, detailed results must be obtained by way of numerical simulation [3–5]. In this paper, samples for the input thermal noise are first generated, and the governing differential equations are solved numerically using the fourth-order Runge-Kutta method. The output samples so obtained are then used to calculate the ensemble averages. The time step in the numerical calculation is chosen according to both the system relaxation time and the time period of the noise intensity.

II. PARTICLE TRANSPORT IN A TEMPERATURE RATCHET

Let the period of the changing temperature field be $\mathcal{T}$, namely,

$$T(t) = T(t+\mathcal{T}). \tag{6}$$

Since the noise intensity is time dependent, the system will never reach a statistically stationary state, i.e., a thermal equilibrium state. However, due to the periodicity of the

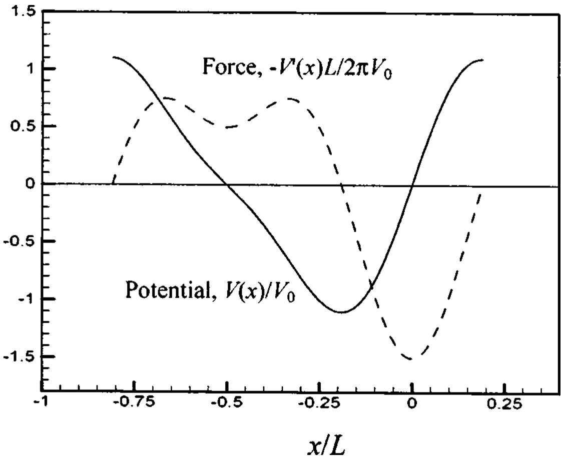

FIG. 1. Potential (9) and corresponding force.

noise intensity, the particle transient mean velocity also tends to be periodic in time, as the effect of the initial state diminishes. The long-term particle current (4) can be calculated from

$$\langle \dot{x} \rangle = \lim_{t\to\infty} \frac{1}{t} \int_0^t E[\dot{x}(t)]dt = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} E[\dot{x}(t)]dt. \tag{7}$$

Consider the following temperature field [1]:

$$T(t) = \overline{T}\{1 + A\ \mathrm{sgn}[\sin(2\pi t/\mathcal{T})]\} \tag{8}$$

where $\overline{T}$ and A are constants, and sgn[ ] denotes the signum function. According to Eq. (8), the system is subjected alternatively to two different noise levels $T=\overline{T}(1-A)$ and $T=\overline{T}(1+A)$. A typical example for the periodic potential is [1,2]

$$V(x) = V_0[\sin(2\pi x/L) + 0.25 \sin(4\pi x/L)]. \tag{9}$$

This potential is illustrated in Fig. 1, as well as the corresponding force $-V'(x)$ within one period. Here a typical period is considered to be between $x_l \approx -0.81L$ and $x_r \approx 0.19L$,

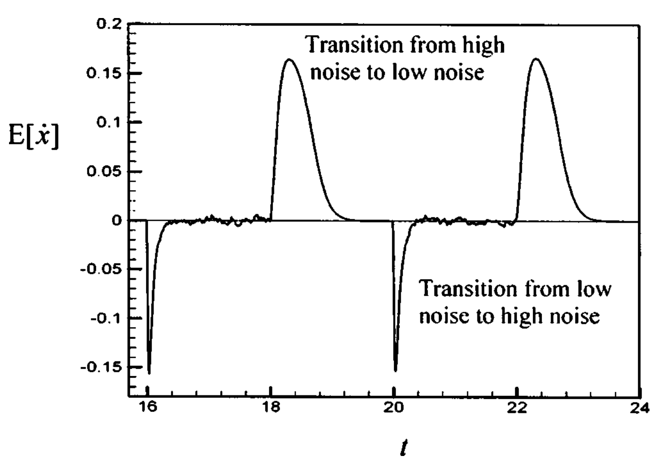

FIG. 2. Transient mean velocity for noise period $\mathcal{T}=4$.

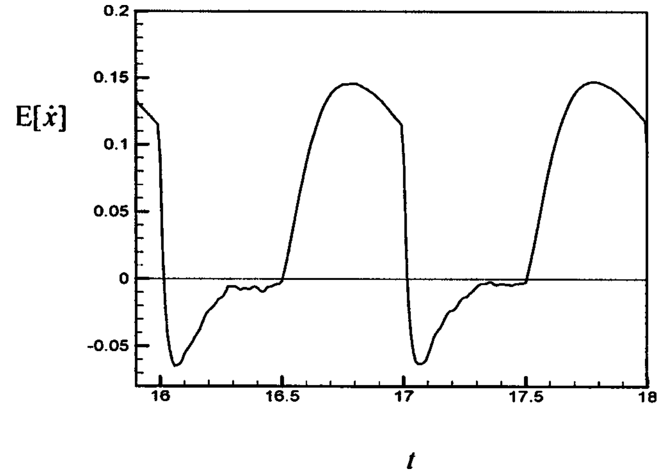

FIG. 3. Transient mean velocity for noise period $\mathcal{T}=1$.

corresponding to two consecutive maxima of $V(x)$. The minimum of $V(x)$ within this period is found at $x_0 \approx -0.19L$.

Numerical calculations were carried out for the temperature ratchet governed by Eqs. (1), (2), (8), and (9) with $\eta=1$, $L=1$, $k_B=1$, $V_0=(1/2)\pi$, $\overline{T}=0.1$, and $A=0.7$, where all parameters are nondimensional and were selected following [1]. Several values were assumed for the noise period $\mathcal{T}$. The transient mean velocity $E[\dot{x}(t)]$ is depicted in Fig. 2 for the case of $\mathcal{T}=4$ and for two periods between $t=16$ and $t=24$. It indeed exhibits a periodic behavior. Since the period $\mathcal{T}$ chosen is long enough, the system transits between two nearly stationary states corresponding to the two noise levels. It is seen that the mean velocity is negative when the noise intensity is increased from a lower level to a higher one and positive when the noise intensity is reversed to the lower one. Moreover, the transition from the lower noise level to the higher level is much shorter than the reverse transition. The two transition processes exhibit a clear asymmetric characteristic, and the difference in the two transition times leads to a positive long-term overall particle current $\langle \dot{x} \rangle$. Figure 3 shows the transient mean velocity for the case of $\mathcal{T}=1$ in two periods after $t=16$. With a shorter period $\mathcal{T}$, the system is

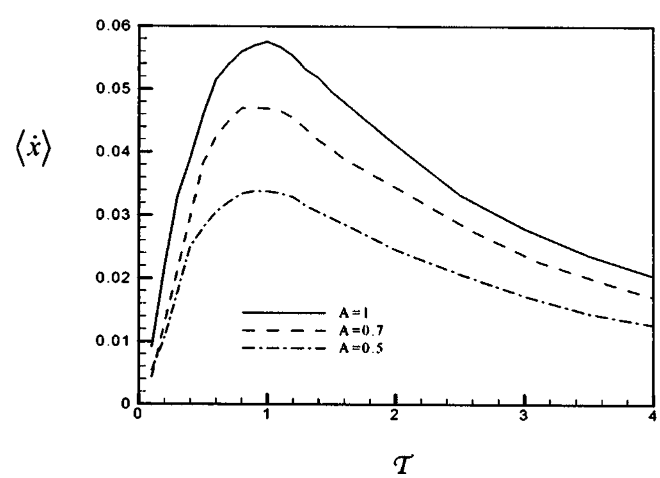

FIG. 4. Long-term particle current vs noise period $\mathcal{T}$ for different values of A.

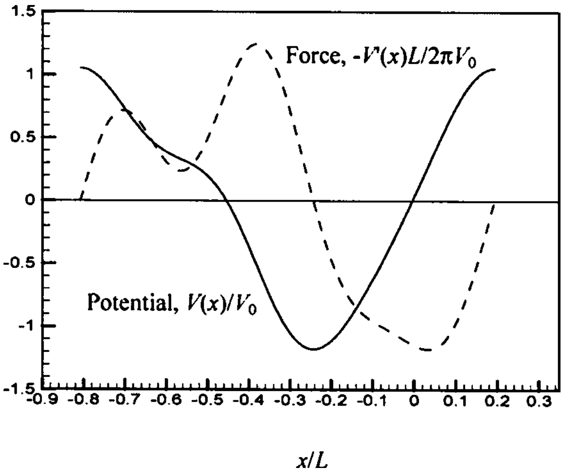

FIG. 5. Potential (10) and corresponding force.

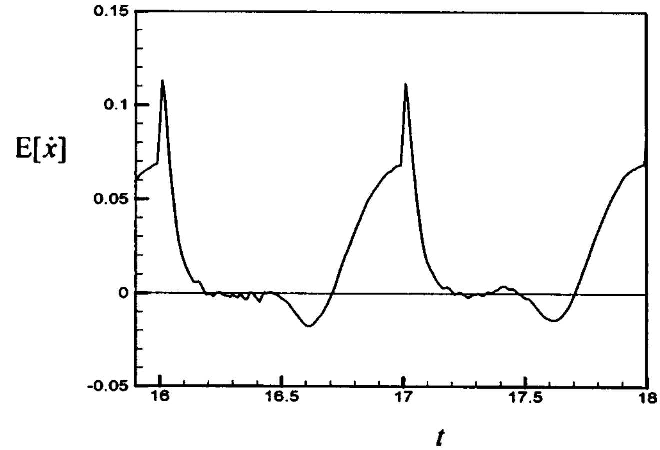

FIG. 7. Transient mean velocity for noise period $\mathcal{T}=1$ and potential (10).

unable to transit between two early stationary states, and the magnitude for both the positive and negative mean velocities also decreases. The long-term particle velocity $\langle \dot{x} \rangle$ is still positive.

The long-term $\langle \dot{x} \rangle$ has been calculated from Eq. (7) for three cases of $A=1$, 0.7, and 0.5, and the results are depicted in Fig. 4. In the case of $A=1$, the system undergoes loading and unloading processes, while in other cases the noise intensity is changed periodically between two different levels. A lower A value indicates a smaller change in the noise intensity, leading to a smaller difference in the two transition periods between two noise levels. As a result, the long-term particle current is also reduced. In every case, the particle current reaches a maximum at a certain $\mathcal{T}$ value. When $\mathcal{T}$ is large and is increasing, the net area under the $E[\dot{x}]$ curve within one period remains the same; thus, the time-averaged value $\langle \dot{x} \rangle$ decreases with an increasing time period. In the limiting case of a very large $\mathcal{T}$, the system approaches a stationary state. On the other hand, if $\mathcal{T}$ is very small, the system does not have enough time to respond to the change of the noise level, leading also to a low long-term particle current. In the entire range of the $\mathcal{T}$ value, the long-term particle current is always in one (positive) direction.

III. CURRENT INVERSION

Now consider another ratchet potential given by [1]

$$V(x) = V_0\left\{ \sin\left(2\pi\frac{x}{L}\right) + 0.2\sin\left[4\pi\left(\frac{x}{L} - 0.45\right)\right] + 0.1\sin\left[6\pi\left(\frac{x}{L} - 0.45\right)\right]\right\}. \quad (10)$$

This potential and the generated force are shown in Fig. 5. The potential has a similar but slightly different asymmetric shape compared to that of Fig. 1, with the two maxima at $x_l \approx -0.80771L$ and $x_r \approx 0.19229L$, and a minimum at $x_0 \approx -0.24349L$. The transient mean velocities $E[\dot{x}]$ of the system, with potential (10) and under the same temperature field (8), are depicted in Figs. 6–9 for several different noise periods $\mathcal{T}$. In the case of $\mathcal{T}=6$, shown in Fig. 6, the system is able to transit between two nearly stationary states, corresponding to the two noise levels. However, the transient behavior is different from that shown in Fig. 2. During the transition from the low noise level to the high one, the mean velocity is first positive and then turns negative until reach-

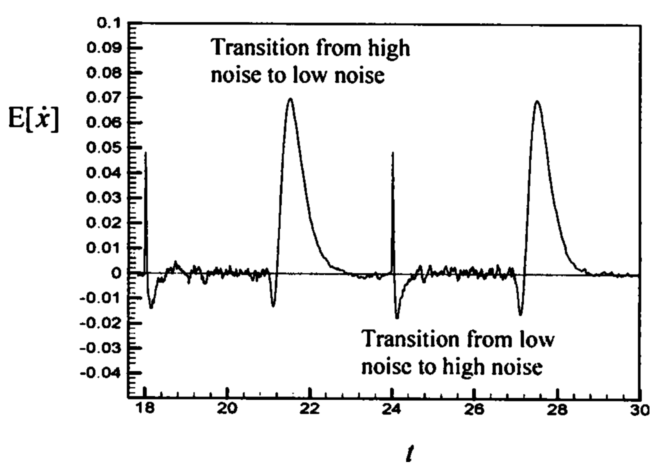

FIG. 6. Transient mean velocity for noise period $\mathcal{T}=6$ and potential (10).

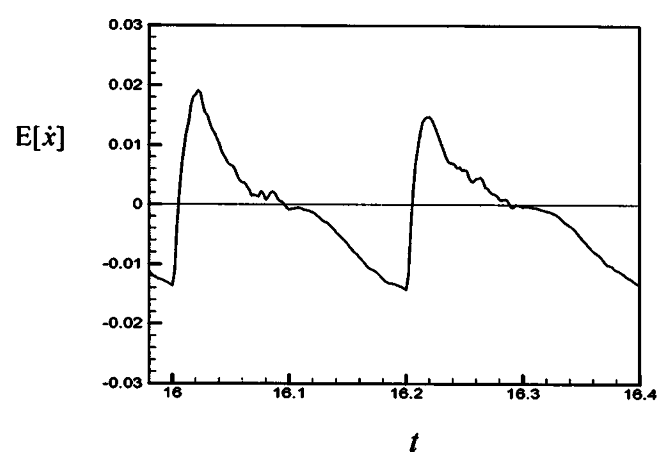

FIG. 8. Transient mean velocity for noise period $\mathcal{T}=0.2$ and potential (10).

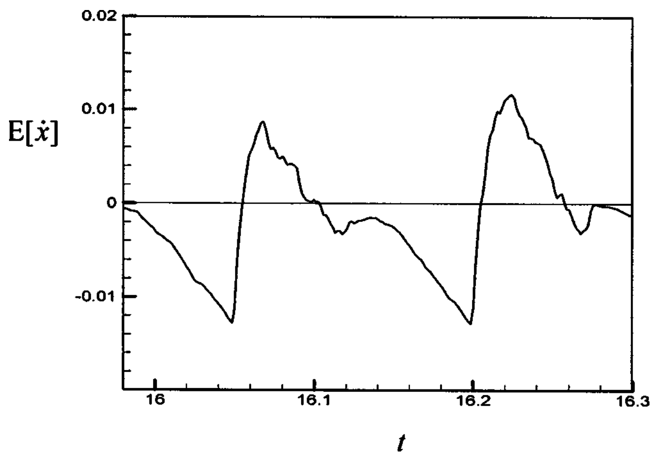

FIG. 9. Transient mean velocity for noise period $\mathcal{T}=0.15$ and potential (10).

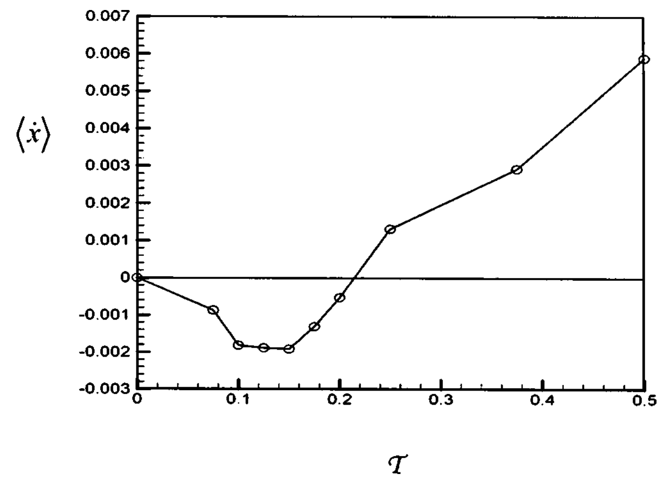

FIG. 10. Current inversion for the case of potential (10).

ing the nearly stationary state. The behavior is reversed during the transition period from the high noise level to the low one. Overall, the long-term time average $\langle \dot{x} \rangle$ during one cycle is positive. However, if the period $\mathcal{T}$ is not long enough, the transition pattern will be broken. Figures 7–9 show the transient mean particle velocity $E[\dot{x}]$ for three $\mathcal{T}$ values of 1, 0.2, and 0.15. It is clear that the long-term particle current $\langle \dot{x} \rangle$ is positive for the case of $\mathcal{T}=1$, and is negative for $\mathcal{T}=0.15$. It is difficult to reach a conclusion just by inspection for the case of $\mathcal{T}=0.2$. The phenomenon that the direction of the particle current changes at a certain parameter value is known as the current inversion [6]. Figure 10 shows the calculated long-term particle current $\langle \dot{x} \rangle$ for different values of period $\mathcal{T}$. It indicates that a current inversion occurs at about $\mathcal{T}=0.21$.

The physical mechanism for the current inversion is not clear. As seen from Figs. 1 and 5, the shapes of the two potentials (9) and (10) are different only in some minor details. The current inversion occurs for the case of potential (10), but not for the case of potential (9). Therefore, the detailed shape of the asymmetric potential is crucial for current inversion.

IV. CONCLUSION

This Brief Report is aimed at explaining the directed transport behavior of temperature ratchets by examining the transient mean particle velocity without additional time averaging. It is found that the transient mean particle velocity may be in one direction during one time period and in the opposite direction during another time period. It is the overall effect, namely, the time average of the mean velocity, that determines the long-term particle current. For cases in which the system statistical behaviors are periodic in time, like the temperature ratchet models investigated in the paper, both the direction and magnitude of the long-term particle current can be predicted by investigating the behavior of the transient mean particle velocity in one period.

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